4.7 Inverse Trigonometric Functions

Recall that in order for a function to have an inverse function, it must be one-to-one (it must pass the Horizontal Line Test). Therefore, in order for the function, $y = \sin x$ to have an inverse, we must restrict the domain.

When restricting the domain to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, the following properties hold.

On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

1. $y = \sin x$ is increasing.
2. The range of $y = \sin x$ is [-1, 1].
3. $y = \sin x$ is one-to-one.

So, restricting the domain of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ gives a unique function called the **inverse sine function** denoted by

$y = \arcsin x$ or $y = \sin^{-1} x$

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$y = \arcsin x$ if and only if $\sin y = x$

where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. The domain of $y = \arcsin x$ is [-1, 1], and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$.

Note that $\sin^{-1} x$ means the inverse sine function and does not mean $\frac{1}{\sin x}$.

The $\arcsin x$ notation is read as "the arcsine of *x*" and comes from the association of a central angle with its intercepted *arc length* on a unit circle. In other words, $\arcsin x$ means the angle (or arc) whose sine is *x*. **Both notations, $\arcsin x$ and $\sin^{-1} x$, mean the same thing and are commonly used in mathematics to find an angle.**

When restricting the domain to $0 \le x \le \pi$, the following properties hold.

On the interval $[0, \pi]$:

4. $y = \cos x$ is decreasing.
5. The range of $y = \cos x$ is [-1, 1].
6. $y = \cos x$ is one-to-one.

So, restricting the domain of $y = \cos x$ to $0 \le x \le \pi$ gives a unique function called the **inverse cosine function** denoted by

$y = \arccos x$ or $y = \cos^{-1} x$

Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Definitions of the Inverse Trigonometric Functions

Function	**Domain**	**Range**
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

In Exercises 1–12, find the exact value.

1. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **2.** $\sin^{-1}\left(-\frac{1}{2}\right)$

3. $\tan^{-1} 0$ **4.** $\cos^{-1} 1$

5. $\cos^{-1}\left(\frac{1}{2}\right)$ **6.** $\tan^{-1} 1$

7. $\tan^{-1}(-1)$ **8.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

9. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **10.** $\tan^{-1}\left(-\sqrt{3}\right)$

11. $\cos^{-1} 0$ **12.** $\sin^{-1} 1$

In Exercises 13–16, use a calculator to find the approximate value. Express your answer in degrees.

13. $\sin^{-1}(0.362)$ **14.** $\arcsin 0.67$

15. $\tan^{-1}(-12.5)$ **16.** $\cos^{-1}(-0.23)$

In Exercises 17–20, use a calculator to find the approximate value. Express your result in radians.

17. $\tan^{-1}(2.37)$ **18.** $\tan^{-1}(22.8)$

19. $\sin^{-1}(-0.46)$ **20.** $\cos^{-1}(-0.853)$

In Exercises 23–32, find the exact value without a calculator.

23. $\cos(\sin^{-1}(1/2))$

24. $\sin(\tan^{-1} 1)$

25. $\sin^{-1}(\cos(\pi/4))$

26. $\cos^{-1}(\cos(7\pi/4))$

27. $\cos(2\sin^{-1}(1/2))$

28. $\sin(\tan^{-1}(-1))$

29. $\arcsin(\cos(\pi/3))$

30. $\arccos(\tan(\pi/4))$

31. $\cos\left(\tan^{-1}\sqrt{3}\right)$

32. $\tan^{-1}(\cos\pi)$

In Exercises 47–52, find an algebraic expression equivalent to the given expression. (*Hint:* Form a right triangle as done in Example 5.)

47. $\sin(\tan^{-1} x)$

48. $\cos(\tan^{-1} x)$

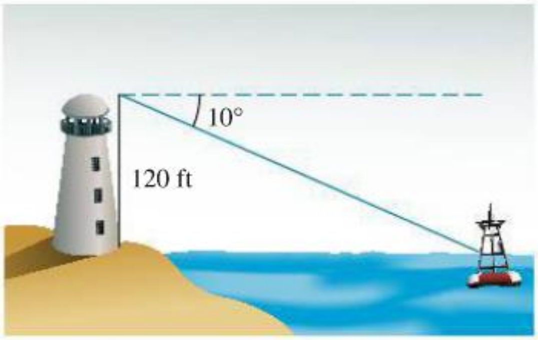
49. $\tan(\arcsin x)$

50. $\cot(\arccos x)$

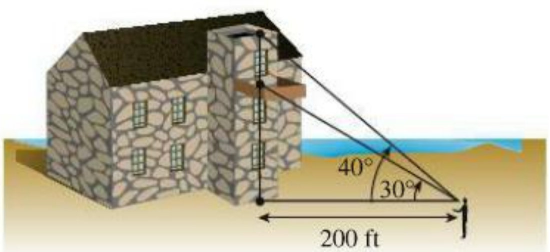
51. $\cos(\arctan 2x)$

52. $\sin(\arccos 3x)$

3. **Finding a Distance** The angle of depression from the top of the Smoketown Lighthouse 120 ft above the surface of the water to a buoy is 10°. How far is the buoy from the lighthouse?



15. **Civil Engineering** The angle of elevation from an observer to the bottom edge of the Delaware River drawbridge observation deck located 200 ft from the observer is 30°. The angle of elevation from the observer to the top of the observation deck is 40°. What is the height of the observation deck?



16. **Traveling Car** From the top of a 100-ft building a man observes a car moving toward him. If the angle of depression of the car changes from 15° to 33° during the period of observation, how far does the car travel?